Is Problem Solving Too Hard?

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Abstract

Problem solving has widely been advocated as the way ahead in the teaching of mathematics. While it is an important development in mathematics education, it may not be as easy to implement as at first thought. We consider four aspects of teaching problem solving to show the difficulties involved. These aspects are the problem solving tasks, the nature of solutions, metacognition, and the scaffolding required to enable students to learn. It is important that teachers wishing to use a problem solving approach in their teaching are aware of all of these aspects of problem solving and, in addition, that professional development is provided in order for them to master the problem solving approach.

Methodology

The conclusions of this paper are based upon two research projects and experience in working with talented mathematics students extending over a period of more than ten years.

The first research project was undertaken as part of the requirements for a PhD, see Thomas (1995). This research (referred to here as the junior mathematics project) examined the interactions that occurred between children ranging from 6 to 8 years old, as they worked independently of the teacher. Among the questions addressed were (i) what is the nature of talk that occurs between children working in groups independent of the teacher in junior mathematics classrooms and (ii) to what extent does this talk help develop mathematical understanding. The second project (referred to as the secondary mathematics project) investigated problem solving in four secondary mathematics classrooms with a view to determining among other things (i) which teachers feel most comfortable teaching problem solving and (ii) which students most benefit from a problem solving approach. In both studies, extensive use was made of video observations, as well as audio taped interviews with the teachers involved.

One of the authors has been involved in problem solving sessions with talented secondary mathematics students since the early 1980s. During these classes, no specific records were taken but there was continual reflection on the students' responses and on the mathematics that they were able to accomplish.

Problem Solving

Problem solving is used, among other things, to mean the process of tackling and solving problems which are new to the person presented with the problem. In this paper, we take the meaning further so that when we use the words **problem solving** we mean **teaching** and **learning** via problem solving. As Schroeder and Lester (1989) say in this context, "problems are valued not only as a purpose for learning mathematics, but also a primary means of doing so. The teaching of a mathematical topic begins with a problem situation that embodies key aspects of the topic, and mathematical techniques are developed as reasonable responses to reasonable problems" (p. 33). (For a detailed discussion of problem solving, see Schoenfeld, 1992.)

In the following section we briefly indicate that teachers have difficulties with the teaching of problem solving. In the final sections we discuss four areas that we believe contribute to this difficulty. These are the nature of the task, handling the types of solution that students produce, using metacognition in the solution of problems, and the scaffolding that is required to lead students from their current knowledge to the satisfactory completion of the task. Finally we suggest approaches to overcome some of these difficulties.

The Problem

The importance of problem solving is now internationally recognised in mathematics education. Perhaps starting in the United States with NCTM (1980) and strengthened by NCTM (1989), it moved to England and Wales with the Cockcroft Report of 1982. In Australia, a number of projects culminated in the Lovitt and Clarke (1988), while in New Zealand, problem solving became mandatory with the gazetting of Mathematics in the New Zealand Curriculum (Ministry of Education, 1992).

However, simply because problem solving has been advocated by educators in a number of countries and has appeared in curriculum documents, it does not mean that problem solving is occurring in the classrooms of these countries. In Australia "All the early problem solving efforts were mostly devoted to the creation of suitable problems in the belief that teachers could present these in classrooms and generate effective learning with the same maths they used for expository teaching. It has taken some time to recognise that this is not the case..." (Lovitt, 1995). In the secondary mathematics project, we observed the same phenomenon.

In retrospect it should not be surprising that teachers have difficulty with this new approach to the teaching and learning of mathematics. Most teachers have had virtually no specific training in using a problem solving approach. It is a well known phenomenon for the "implemented" (actual classroom) curriculum, to lag behind the "intended" (prescribed) curriculum (Kilpatrick, 1995). There is no reason for problem solving to be any different in this regard.

The Nature of the Task

By task here, we mean the activity around which problem solving and the learning of mathematics, takes place. As proposed in Thomas (1995), the effectiveness of the task in stimulating learning is a function of the cognitive and cooperative demands of the activity. Other dimensions of the task which may have an impact on learning include the context and wording of the problem.

The **cognitive** aspect is clear. If the problem is too easy, there will be no problem solving required. If the problem is too difficult, the students may be unable to make any progress on it. The importance of matching the cognitive challenge of the task to the capabilities of the students is consistent with Vygotsky's notion of a zone of proximal development (Vygotsky, 1962).

It is within the context of providing appropriately challenging tasks that the concept of rich mathematical activities have been proposed. These are activities which may be tackled at a variety of levels by a range of students. Each student can find a cognitive level in the problem which will provide a challenge for her/him. One of the goals of the Mathematics Curriculum and Teaching Program, was to collect and disseminate rich mathematical activities. Other sources include Bird (1986), Gardiner (1987), Lowe and Lovitt (1984), and Stacey and Groves (1985).

Tasks may or may not require **cooperation**. Variations in the cooperative demands of the task place different participation demands on the students. Thomas (1995), reports that the high organisational aspects of some tasks reduces the focus on the problem solving itself. Her research with junior mathematics classes suggests that reducing the size of groups to two led to less social talk and more talk about the cognitive aspects of the task. The secondary mathematics project, found that groups of four, frequently split into smaller groups when tackling assigned problems. There is considerable anecdotal evidence from teachers to support this. This project also observed less concerns over social issues with the 13-year-olds in their study than occurred with the 8-year-olds in the primary mathematics project. This suggests that cooperation is easier to manage by older students.

The next aspect of the task that we want to mention is the **contextual** dimension. Much has been said in the literature about problem contexts which cause difficulty based on the gender or culture of the students (see, for example, Clark, 1994). In the secondary mathematics project we noted that the context of the problem affects the enthusiasm with which the students approach the problem. The junior mathematics project teachers had the most success with problems which were linked directly to the current interests of the students.

The actual wording of a problem can change the difficulty for the student. Stacey's MATCH task (Stacey, 1994) is a good example. The students are given a drawing of a match, and underneath they are told:

The match is 2 cm shorter than the line. Draw the line.

In her research, many students drew the line shorter than the match. While MATCH is perhaps at first sight, an example of language being used an unusual way, it is likely that teachers will often use language and write problems in a way that is unusual from their students perspective. (See also Siemon, 1993.)

Types of Solution

Problem solving is promoted in some areas because it provides opportunities to tackle problems which have "more than one solution" (Ministry of Education, 1992). In this section we consider what that means precisely, and consider the types of solution proposed by Holton and Daniel (1995).

In this context, by the **answer** to a problem we mean the number or result that is the final outcome of the attack on this problem. By the **method** of a problem, we mean the manner by which the answer is obtained. The **solution** of a problem is the sum total of the method and the answer.

Holton and Daniel (1995), recognise a number of types of solution that a problem may have. These are listed below in increasing order of sophistication. Naturally any given problem may be solvable by more than one type of solution. In addition, there may be more than one solution to a given problem which is of a given type.

1. answer incorrect; method incorrect;
2. no answer or answer incorrect; method unclear;
3. answer correct; method inadequately described;
4. answer incorrect; method correct;
5. answer correct; method correct but elementary or laborious;
6. answer correct; method correct using standard mathematics;
7. answer correct; method correct using sophisticated ideas;
8. generalization or extension provided to the original problem.

The reason that the existence of types of solution provides difficulties for teachers lies in the fact that when approaching a student or group of students, the teacher has to (i) assess the solution type being pursued by the student; (ii) decide whether the method is valid; (iii) decide whether or not the student needs assistance; (iv) determine the best way to provide assistance; (v) decide when to move on.

If the teacher has a particular solution in mind, it is not always easy to recognise whether the student working on a different approach has a valid method. It is easy to misunderstand and suggest that a student abandons a potentially fruitful approach. The teacher also needs to assess the cognitive level that a student can achieve. Should the teacher be satisfied with a type (5) solution or encourage the student to reach (6) or even (7)?

Metacognition

From the work of Schoenfeld, (1992), we know that good problem solvers use **metacognition** - thinking about thinking and using this to control progress through a problem. Schoenfeld says that metacognition includes an individual's declarative knowledge about their cognitive process; self-regularity procedures, including monitoring and "on-line" decision-making; and beliefs and affects and their effects on performance. This metacognition leads problem solvers to control the heuristics that they apply to a problem and helps them to decide, among other things, when a given line of inquiry is proving unfruitful and should be abandoned.

This cognitive process has great importance. It is not one that regularly comes into play in the traditional classroom setting. If the problem being solved is able to be completed in a short period, then there is no need to monitor progress or think about and choose between, a range of solution strategies. The same is true if the students are practising an algorithm that they have just been taught. Any reasonable problem solving situation however, is likely to involve an investment of a considerable amount of time. In this circumstance, a range of questions such as the following are invaluable for keeping track of progress and for keeping on track toward a solution.

What (exactly) are you doing? Can you describe it precisely?

Why are you doing it? How does it fit into the solution?

How does it help you? What will you do with the outcome when you obtain it?

Have I been trying this approach for too long! Should I try something else?

When the students finally get an answer it is also worth asking:

Does the answer make sense? Is it consistent with previous knowledge?

What questions can we ask now?

What generalisations or extensions are worth following up?

Unfortunately we seem to know little about metacognition. How does it develop as we grow older? Can metacognition be taught?

A deeper knowledge of metacognition would certainly aid the teaching and learning of problem solving. Even so it is something of which teachers need to be aware. By inserting metacognitive comments into their discussions with students, students will come to appreciate the value of the process. Hopefully they will gradually learn its advantages for them and begin to put it into practice, at least in mathematical problem solving sessions.

Scaffolding

The process of helping students from their current state of knowledge within their zone of proximal development is called **scaffolding**. The concept was introduced by Vygotsky (1962). Greenfield (1984, p. 118) describes it as follows:

> *The scaffold is a metaphor to describe the ideal role of the teacher. The scaffold, as it is known in building construction, has five characteristics: it provides a support; it functions as a tool; it extends the range of the worker; it allows the worker to accomplish a task not otherwise possible; and it is used selectively to aid the worker where needed...*

These characteristics also define the interactional scaffold provided by the teacher in a learning situation. That is, the teacher's selective intervention provides a supportive tool for the learner, which extends his or her skills, thereby allowing the learner to successfully accomplish a task not otherwise possible. Put another way, the teacher structures an interaction by building on what he or she knows the learner can do. Scaffolding thus closes the gap between task requirements and the skill level of the learner."

Bickmore-Brand and Gawned (1990, p. 54) note

In order to achieve effective scaffolding of mathematical activities the scaffolding must be tailored to the individual needs of the child. The scaffolding must be provided at any point during the task when a shared focus might be seen as beneficial for the child. The adult scaffolding should consist of a blend of focus questions interspersed with comments, information, suggestions and modelling of metacognitive language and the language of the task."

The basis of good scaffolding is to enable the student to produce the solutions for **themselves**. The teacher should provide **open** questions that will lead the student to the solution.

Scaffolding is a process carried out largely by the teacher, though it can be done by other students. Although its primary aim is to link the student with the learning, the method used is likely to be internalised by the student and so accelerate learning in later situations. As such, the scaffolding provided by the teacher is very important in developing habits of learning in the student.

Good scaffolding requires not only a thorough knowledge of the task, its nature, the types of solution possible and metacognition but also a thorough knowledge of the students and their current knowledge. So it is a non-trivial process. If the class has been divided into groups, then teachers must quickly assess each group situation as they approach it and make split-second decisions as to what scaffolding is required. Good scaffolding is an art which takes time to perfect. Even experienced teachers can get it wrong.

Scaffolding comes in roughly five stages. The first stage occurs while students are grappling with the problem to try to understand it. This stage may require reasonably long silences on the part of the teacher while the students come to grips with what the question is asking. Research has show that teachers are not comfortable with long waiting times (see Clarke, 1992). However, the initial stage in solving a problem, especially a complicated one, is to understand the problem. Students need to be able to make their own sense of a task before they can make progress. Hence in the initial stage of scaffolding, teachers must ensure that the task is understood. But as part of this they may need to give students some space. This may mean quiet observation on the part of the teacher for at least sixty seconds.

The second stage of scaffolding is the assessment of where the students have reached in their work. This can be discovered by looking at their work, by asking how they are going and what problems they are facing. If everything appears to be moving along smoothly, the teacher should withdraw. When there is a student impasse, the teacher will need to ask questions such as

What have you tried?

Why did those approaches fail?

What else could you try?

Have you seen a problem like this before?

Do you need to try some more experiments?

Have you recorded your experiments tidily?

Note that these questions are open questions designed to lead a student forward. Scaffolding is **not** about making statements ("use quadratic equations" is directing students). Scaffolding is not about closed questions.

Having determined where the student is and what difficulties they are having the third stage of scaffolding is to produce questions which enable the student to move along, step by step to a solution. Such questions will be problem specific but go along the lines of

Do you know where you are heading?

What intermediate position would enable you to get there?

How could you reach that intermediate position?

In the fourth stage of scaffolding, students have reached a solution. Because there may be more than one solution method, some of which may be better than others, it is worth exploring questions such as

Can you shorten your argument?

Could you use another approach?

Which way of solving this problem is best? Why?

Finally, scaffolding can be used to tempt students into generalisations or extensions.

Can you think of a problem similar to this one?

What parts of the problem could we change to produce a new problem?

Is that new problem likely to be interesting?

In what has become the traditional classroom where students are working on routine problems, scaffolding isn't required to any great extent. It is therefore a skill that teachers have generally not met in their own mathematical learning. However, it is an important integral part of problem solving, and a skill that has to be developed if problem solving is to be an essential part of the mathematics classroom.

Concluding Comments

The four aspects of problem solving discussed here, nature of the task, scaffolding, types of solution and metacognition, along with the lack of problem solving experience on the part of many teachers, all make problem solving in the classroom a difficult venture. If problem solving is to be incorporated into the mathematics curriculum, the various components of problem solving must be recognised. To assist in the development of problem solving, it needs to become an integral part of both pre-service and in-service programmes.

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